

The hat problem on a directed graph

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Abstract

A team of players plays the following game. After a strategy session, each player is randomly fitted with a blue or red hat. Then, without further communication, everybody can try to guess simultaneously his or her own hat color by looking at the hat colors of other players. Visibility is defined by a directed graph; that is, vertices correspond to players, and a player can see each player to whom she or he is connected by an arc. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The team aims to maximize the probability of a win, and this maximum is called the hat number of the graph.

Previous works focused on the problem on complete graphs and on undirected graphs. Some cases were solved, e.g., complete graphs of certain orders, trees, cycles, bipartite graphs. These led Uriel Feige to conjecture that the hat number of any graph is equal to the hat number of its maximum clique.

We show that the conjecture does not hold for directed graphs, and build, for any fixed clique number, a family of directed graphs of asymptotically optimal hat number. We also determine the hat number of tournaments to be one half.

Keywords: hat problem, directed graph, skeleton, clique number.

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1 Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly and independently placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning.

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Origin. The hat problem with seven players, called the “seven prisoners puzzle”, was formulated by Todd Ebert in his Ph. D. Thesis [6]. It is often posed as a puzzle (e.g., in the Berkeley Riddles [2]) and was also the subject of articles in the popular media [3, 15, 16].

The hat problem with $q \geq 2$ possible colors was investigated in [14]. Alon [1] proved that the q -ary hat number of the complete graph tends to one as the graph grows.

Many other variations of the problems exist, among them a random but non-uniform hat color distribution [10], an adversarial allocation of hat from a pool known by the players [8], a variation in which passing is not allowed [4], and many more.

Our focus. We consider the hat problem on a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. We seek to determine the hat number of the graph, that is, the maximal chance of success for the hat problem in it. This variation of the hat problem was first considered in [11].

Note that the hat problem on the complete graph is equivalent to the original hat problem. This case was solved for $2^k - 1$ players in [7] and for 2^k players in [5]. In [14] it was shown that a strategy for n players in the complete graph is equivalent to a covering code of radius 1 in the Hamming cube.

The hat problem was solved for trees [11], cycles [9, 12, 13], bipartite graphs [9], perfect graphs [9], and planar graphs containing a triangle [9]. Feige [9] conjectured that for any graph the hat number is equal to the hat number of its maximum clique. He proved this for graphs with clique number $2^k - 1$. The simplest remaining open case is thus triangle-free graphs.

In this paper we consider the hat problem on directed graphs. Under an appropriate definition of the clique number for directed graphs, we construct families of digraphs with a fixed clique number the hat number of which is asymptotically optimal.

2 Preliminaries

We begin with some definitions regarding directed graphs (digraphs) and undirected graphs.

Definition 2.1. The *skeleton* of a digraph $D = (V, A)$, denoted by $\text{skel}(D)$, is the undirected graph on the vertex set V in which x and y are adjacent if both arcs between them belong to the set A ; that is, if they form a directed 2-cycle in D .

Definition 2.2. The *clique number* of a digraph D is the clique number of its skeleton; that is, $\omega(D) = \omega(\text{skel}(D))$.

Definition 2.3. The *transpose* of a digraph $D = (V, A)$ is the digraph $D^t = (V, A^t)$, where $A^t = \{(x, y) : (y, x) \in A\}$.

Slightly abusing notation, we identify a digraph D with its (undirected) skeleton in the case that $D = D^t$; that is, if all arcs of D have anti-parallel counterparts.

Fix a digraph $D = (V, A)$ on the vertex set $V = \{v_1, v_2, \dots, v_n\}$. We proceed with a more precise definition of the hat problem on D .

Definition 2.4. A (hat) configuration is a function $c : V \rightarrow \{\text{blue}, \text{red}\}$, assigning the hat color $c(v)$ to the vertex $v \in V$. Naturally, there are 2^n possible configurations.

Definition 2.5. The view of a vertex $v \in V$ of a configuration $c : V \rightarrow \{\text{blue}, \text{red}\}$ is the restriction of c to vertices seen by v , namely the function $c^v = c|_{N^+(v)}$. Since the domain of c^v is $N^+(v)$, a set of size $d^+(v)$, the number of possible views for v is $2^{d^+(v)}$. Note that $2^{n-d^+(v)}$ different configurations share any single view of v .

Sometimes we will regard configurations and views as binary vectors of the respective length; that is, $c \in \{\text{blue}, \text{red}\}^n$ and $c^v \in \{\text{blue}, \text{red}\}^{d^+(v)}$.

Definition 2.6. An individual strategy for the vertex $v \in V$ is a function mapping views to guesses; that is, $g^v : \{\text{blue}, \text{red}\}^{d^+(v)} \rightarrow \{\text{blue}, \text{red}, \text{pass}\}$. A (team) strategy is a sequence $\mathcal{S} = (g^1, \dots, g^n)$ of n individual strategies, where g^i is a strategy for v_i .

Definition 2.7. For a configuration $c \in \{\text{blue}, \text{red}\}^n$ and an individual strategy g^v for a vertex $v \in V$, we say that v guesses correctly if $g^v(c^v) = c(v)$ and guesses wrong if $g^v(c^v) \notin \{\text{pass}, c(v)\}$. For a configuration $c \in \{\text{blue}, \text{red}\}^n$ and a strategy \mathcal{S} , we say that the team wins if at least one vertex guesses correctly and no vertex guesses wrong.

Definition 2.8. The chance of success $\mathbb{P}(\mathcal{S})$ of a strategy \mathcal{S} is the probability that the team wins, using \mathcal{S} , at a configuration selected uniformly at random from $\{\text{blue}, \text{red}\}^n$. The hat number of the digraph D is the maximum $h(D) = \max_{\mathcal{S}} \mathbb{P}(\mathcal{S})$. A strategy \mathcal{S} is optimal for D if $\mathbb{P}(\mathcal{S}) = h(D)$.

By solving the hat problem on a digraph D we mean finding $h(D)$.

The hat problem on undirected graphs was treated in [9, 11]. We now cite four claims that generalize to digraphs with little or no change.

Claim 2.9. For every two digraphs D and E such that $D \subseteq E$ we have $h(D) \leq h(E)$.

Claim 2.10. For every digraph D we have $h(D) \geq 1/2$.

Claim 2.11. Let D be a digraph and let v be a vertex of D . If \mathcal{S} is a strategy for D in which v always attempts to guess its color, then $\mathbb{P}(\mathcal{S}) \leq 1/2$.

Claim 2.12. Let D be a digraph and let v be a vertex of D . If \mathcal{S} is an optimal strategy for D in which v never attempts to guess its color, then $h(D) = h(D - v)$.

Combining Claims 2.10, 2.11 and 2.12 we get the following.

Claim 2.13. Let D be a digraph and let v be a vertex of D . If v has no outgoing arcs, i.e., $d^+(v) = 0$, then $h(D) = h(D - v)$.

3 Constructions

For an undirected graph G , it is known that if G contains a triangle, then $h(G) \geq 3/4$, and it is conjectured in [9] that if G is triangle-free, then $h(G) = 1/2$. Do directed graphs introduce anything in between? The answer is yes.

Let us consider the hat problem on the digraph D_1 given in Figure 1.

Fact 3.1. $h(D_1) = 5/8$.

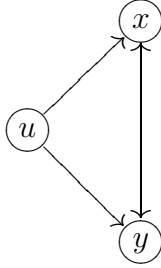


Figure 1: The directed graph D_1

We omit the proof of this fact in favor of extending D_1 to a construction of a family $\{D_n\}_{n=0}^\infty$ of semi-complete digraphs that asymptotically achieve hat number $2/3$, with the property that the $\omega(D_n) = 2$.¹

Definition 3.2. Given two disjoint digraphs C and D , we define the *directed union* of C and D , denoted by $C \rightarrow D$, as the disjoint union of these two digraphs with the additional arcs from all vertices of C to all vertices of D . Note that this operator is associative; that is, $C \rightarrow (D \rightarrow E) = (C \rightarrow D) \rightarrow E$ for any three digraphs C , D and E . Thus, the notation $C \rightarrow D \rightarrow E$ is unambiguous. We denote the directed union of n disjoint copies of a digraph D by $D^{\rightarrow n} = \underbrace{D \rightarrow D \rightarrow \cdots \rightarrow D}_n$.

Expressed in the terms of directed union, $D_1 = K_1 \rightarrow K_2$. We extend this to a family of digraphs by defining $D_n = K_1 \rightarrow K_2^{\rightarrow n}$. Note that the family $\{D_n\}_{n=0}^\infty$ satisfies the recurrence relation $D_{n+1} = D_n \rightarrow K_2$ for $n \in \mathbb{N}$.

In Figure 2 we give examples of D_n for $n = 2$, $n = 3$, and a general n .

We proceed to compute the hat number of the digraphs of the family $\{D_n\}_{n=0}^\infty$. First, we prove the upper bound.

Lemma 3.3. *For any digraph D we have $h(D \rightarrow K_2) \leq \max\{h(D), 1/2 + (1/4)h(D)\}$.*

Proof. Let \mathcal{S} be a strategy for $D \rightarrow K_2$. Denote the K_2 vertices by x and y , and let us consider the sub-strategy played by x and y .

¹Moreover, the skeleton of D_n is a matching of size n plus an isolated vertex. For short, we write $\text{skel}(D_n) = nK_2 \cup K_1$.

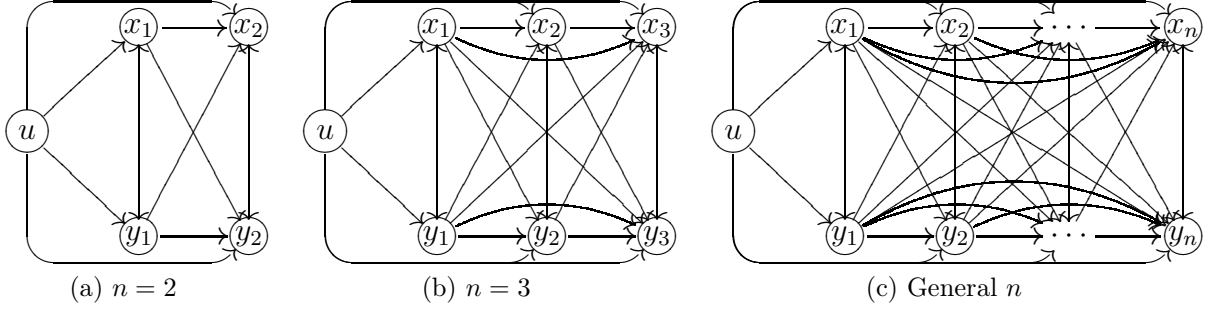


Figure 2: The directed, semi-complete graphs D_2 , D_3 , and D_n . All vertical arcs have anti-parallel counterparts. The remaining arcs are rightwards

Case 1. If at least one of x and y always tries to guess, then $\mathbb{P}(\mathcal{S}) \leq 1/2$.

Case 2. If at least one of x and y never guesses its color, without loss of generality let it be y . Then, by Claims 2.12 and 2.13 we have $\mathbb{P}(\mathcal{S}) \leq h(D \rightarrow K_2 - y) = h(D \rightarrow K_1) = h(D)$.

Case 3. If both x and y guess their colors sometime, then each one guesses its color with probability $1/2$ as every one of them has just one outgoing arc. Hence, with probability at least $1/4$ at least one is wrong. The chance of success of the strategy \mathcal{S} benefits from the behavior of the vertices of D only when both x and y pass, and this happens exactly with probability $1/4$ since they see different vertices (that is, each other). Since the behavior of the vertices of D when both x and y pass is a strategy \mathcal{S}' for D , we can bound

$$\mathbb{P}(\mathcal{S}) \leq 1/2 + (1/4)\mathbb{P}(\mathcal{S}') \leq 1/2 + (1/4)h(D).$$

The result is established by taking \mathcal{S} to be an optimal strategy for $D \rightarrow K_2$. \square

The next lemma proves the lower bound in a more general setting.

Lemma 3.4. *For every positive integer m there exists $c > 0$ such that for any digraph D we have $h(D \rightarrow K_m) \geq cm/(m+1) + (1-c)h(D)$.*

Proof. Let \mathcal{S} be an optimal strategy for the digraph D . We describe a strategy \mathcal{S}' for the digraph $D \rightarrow K_m$. Denote the vertices of K_m by x_1, x_2, \dots, x_m .

1. The vertices of D pass if at most one of $\{x_1, \dots, x_m\}$ has a red hat; otherwise, they behave according to the strategy \mathcal{S} .
2. For $i = 1, \dots, m$, the vertex x_i can see the $m-1$ vertices $\{x_j : j \neq i\}$. If all of them have blue hats, then x_i guesses red; otherwise it passes.

If x_1, \dots, x_m all have a blue hat, then they all guess wrong. If exactly one of them, x_i , had a red hat, then x_i guesses correctly and all other vertices pass. All in all, conditioned on

the event \mathcal{A} that at most one of x_1, \dots, x_m has a red hat, the team wins with probability $m/(m+1)$. Let $c = \mathbb{P}(\mathcal{A}) = \mathbb{P}(\text{Bin}(m, 1/2) \leq 2) = (m+1)2^{-m}$. We have

$$\mathbb{P}(\mathcal{S}') = \mathbb{P}(\mathcal{A})m/(m+1) + (1 - \mathbb{P}(\mathcal{A}))\mathbb{P}(\mathcal{S}) = cm/(m+1) + (1-c)h(D),$$

establishing the result. \square

Remark. In the proof of Lemma 3.4, c approaches zero very quickly as m grows. In fact, we can have $c \geq 1/2$ by using a slightly more complicated strategy. Let $C \subset \{\text{blue}, \text{red}\}^m$ be a code of distance 3, and consider the packing of stars $K_{1,m}$ in the hypercube graph H_m formed by selecting balls of radius one around each codeword.

The event \mathcal{A} is now defined as the event that the configuration of x_1, \dots, x_m is covered by the packing. Step 1 stays basically the same: the vertices of D all pass if \mathcal{A} occurred and behave according to \mathcal{S} otherwise. Step 2 is generalized to make use of the entire packing: if x_i sees a configuration consistent with some codeword, it guesses the color that disagrees with it. As before, when \mathcal{A} occurs either m vertices guess wrong or exactly one guesses, and is correct.

Now the existence of codes of distance 3, length m , and size $\lceil 2^{m-1}/(m+1) \rceil$ implies that $c \geq 1/2$.

We use Lemmata 3.3 and 3.4 to calculate the hat number of D_n .

Corollary 3.5. *For every non-negative integer n we have*

$$h(D_n) = \frac{2}{3} - \frac{1}{6} \cdot \frac{1}{4^n}.$$

Proof. We prove the result by induction on the number n . For $n = 0$ the claim is obviously true as D_0 is a single isolated vertex and $h(D_0) = 1/2 = 2/3 - 1/6$. Let n be a positive integer, and assume that $h(D_{n-1}) = 2/3 - 1/4^{n-1}$. Since $h(D_{n-1}) < 2/3$, by Lemma 3.3 we have

$$h(D_n) \leq \max\{h(D_{n-1}), 1/2 + (1/4)h(D_{n-1})\} = 1/2 + (1/4)h(D_{n-1}).$$

This is matched by Lemma 3.4, which gives $h(D_n) \geq (3/4)(2/3) + (1/4)h(D_{n-1})$. Therefore

$$h(D_n) = \frac{1}{2} + \frac{1}{4}h(D_{n-1}) = \frac{1}{2} + \frac{1}{4} \left(\frac{2}{3} - \frac{1}{6} \cdot \frac{1}{4^{n-1}} \right) = \frac{2}{3} - \frac{1}{6} \cdot \frac{1}{4^n}$$

and the result is established. \square

We have just proved the following.

Theorem 3.6. *For every $\varepsilon > 0$ there exists a digraph D satisfying $\omega(D) = 2$ such that $h(D) > 2/3 - \varepsilon$.*

Theorem 3.6 can be generalized to an arbitrary clique number m .

Theorem 3.7. *For every $\varepsilon > 0$ there exists a digraph D satisfying $\omega(D) = m$ such that $h(D) > m/(m+1) - \varepsilon$.*

Proof. Let us consider $D = K_m^{\rightarrow n}$, where $n = \lceil \log_{1-c}(\varepsilon) \rceil$ and c is the appropriate constant from Lemma 3.4. By repeatedly applying the lemma we get that

$$h(D) \geq (1 - (1 - c)^n) m / (m + 1) \geq (1 - \varepsilon) m / (m + 1) > m / (m + 1) - \varepsilon,$$

as needed. □

The natural question to ask is whether $m/(m+1)$ is the best possible hat number of such digraphs. In the following section we show that indeed this is the best possible, i.e., that the chance of success $m/(m+1)$ is asymptotically optimal for digraphs with clique number m .

4 The upper bound

Feige [9] proved that for every undirected graph G we have $h(G) \leq \omega(G) / (\omega(G) + 1)$. We repeat his proof, refining it a bit to show that the same holds for digraphs.

Proposition 4.1. *For every digraph D we have $h(D) \leq \omega(D) / (\omega(D) + 1)$.*

Proof. Let \mathcal{S} be an optimal strategy for D . By $W_{\mathcal{S}}$ let us denote the set of configurations in which the team wins using the strategy \mathcal{S} and by $L_{\mathcal{S}}$ let us denote the set of configurations in which the team actively loses using the strategy \mathcal{S} , that is, configurations in which \mathcal{S} causes at least one wrong guess.

We define a bipartite graph B whose left-hand side is $L_{\mathcal{S}}$, and right-hand side is $W_{\mathcal{S}}$. A losing configuration $l \in L_{\mathcal{S}}$ is adjacent to a winning configuration $w \in W_{\mathcal{S}}$ if they differ only by one coordinate, which is the hat color of a vertex $v \in V(G)$ that attempted to guess at these configurations.² Let us examine the right and the left degrees in B .

Right degree. Let $w \in W_{\mathcal{S}}$ be a winning configuration, and let $v \in V(D)$ be a vertex that guesses correctly at w . Let l be a hat configuration identical to w except in coordinate v . Since v does not see any difference between w and l , it makes the same guess in l , but now it is incorrect.

Therefore $l \in L_{\mathcal{S}}$ is a neighbor of w in B , and $d(w) \geq 1$.

Left degree. Let $l \in L_{\mathcal{S}}$ be a losing configuration, and let $w_1, \dots, w_d \in W_{\mathcal{S}}$ be its neighbors in B , where $d = d(l)$. For every $i = 1, \dots, d$ let $v_i \in V(D)$ be the coordinate at which l and w_i differ.

Assume for the sake of contradiction that some arc $v_i \rightarrow v_j$ is not present in D . By the definition of v_i , it makes a correct guess at the configuration w_i . It cannot tell w_i apart from

²Since v cannot see its own hat color, it acts the same in both hat configurations l and w .

l , and thus it makes the same, now wrong, guess at the configuration l . But then it must make the same incorrect guess at the configuration w_j , which only differs from l by the color of v_j , unseen by v_i . This contradicts the fact that w_j is a winning configuration.

Therefore $\{v_i\}_{i=1}^d$ is a clique in $\text{skel}(D)$ and $d = d(l) \leq \omega(\text{skel}(D)) = \omega(D)$.

We have shown that the right degree in B is at least one and the left degree in B is at most $\omega(D)$. This implies that $|W_S| \leq |E(B)| \leq \omega(D)|L_S|$ and consequently

$$h(D) = \mathbb{P}(\mathcal{S}) = |W_S| \cdot 2^{-|V(D)|} \leq |W_S| / (|W_S| + |L_S|) \leq \omega(D) / (\omega(D) + 1),$$

establishing the result. \square

Remark. Observe that for a digraph D , the hat number $h(D)$ is always a rational number whose denominator is a power of two. Thus, $h(D) < \omega(D) / (\omega(D) + 1)$ unless $\omega(D) + 1$ is a power of two.³

Corollary 4.2. *For every tournament T we have $h(T) = 1/2$.*

Proof. Apply Proposition 4.1 with $\omega(T) = 1$. The lower bound is by Claim 2.10. \square

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³When $\omega(D) + 1 = 2^k$ is a power of two, the upper bound is met by a complete graph K_{2^k-1} as $h(K_{2^k-1}) = (2^k - 1) / 2^k$.

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